

Contraction in L^1 and large time behavior for a system arising in chemical reactions and molecular motors

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Abstract

We prove a contraction in L^1 property for the solutions of a nonlinear reaction–diffusion system whose special cases include intercellular transport as well as reversible chemical reactions. Assuming the existence of stationary solutions we show that the solutions stabilize as t tends to infinity. Moreover, in the special case of linear reaction terms, we prove the existence and the uniqueness (up to a multiplicative constant) of the stationary solution.

Key words: weakly coupled system, molecular motor, transport, parabolic systems, contraction property.

AMS subject classification: 34D23, 35K45, 35K50, 35K55, 35K57, 92C37, 92C45.

1 Introduction

We start with two specific reaction-diffusion systems. The first one describes a reversible reaction and the other one a molecular motor. We first consider the reversible chemical reaction (see also Bothe [4], Bothe and Hilhorst [5], Desvillettes and Fellner [10] and Érdi and Tóth [11]). It involves a reaction-diffusion system of the form

$$\begin{aligned} u_t &= d_1 \Delta u - \alpha k(r_A(u) - r_B(v)) & \text{in } \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^d, \\ v_t &= d_2 \Delta v + \beta k(r_A(u) - r_B(v)) & \text{in } \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^d, \end{aligned} \quad (1.1)$$

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together with the homogeneous Neumann boundary conditions, where $d_1, d_2, \alpha, \beta, k$ and T are positive constants and where Ω is a bounded subset of \mathbb{R}^d with smooth boundary. Such systems describe, with a suitable choice of the functions r_A and r_B , chemical reactions for two mobile species. For example, functions $r_A(u) = u^k, r_B(v) = v^m$ correspond to a reversible reaction $kA \rightleftharpoons mB$. Reactions of the type $q_1A_1 + \dots q_kA_k \rightleftharpoons q_1B_1 + \dots q_mB_m$ can also be described by similar systems with more complicated reactions terms.

Another model problem is a system in $d = 1$ space dimension and n unknown variables $u_1, \dots, u_n, n > 1$, for intercellular transport, namely

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \frac{\partial}{\partial x} \left(\sigma \frac{\partial u_i}{\partial x} + u_i \psi'_i \right) \\ &\quad + \sum_{j=1}^n a_{ij} u_j \quad \text{in } Q_T = [0, 1] \times (0, T) \\ \sigma \frac{\partial u_i}{\partial x} + u_i \psi'_i &= 0 \quad \text{on } \partial Q_T = \{0, 1\} \times (0, T), \end{aligned}$$

where

$$\begin{aligned} a_{ii} &\leq 0, \quad a_{ij} \geq 0 \quad \text{for all } i \in \{1, \dots, n\}, i \neq j, \\ \sum_{i=1}^n a_{ij} &= 0 \quad \text{for all } i, j \in \{1, \dots, n\}. \end{aligned} \tag{1.2}$$

It models transport via motor proteins in the eukaryotic cell where chemical energy is transduced into directed motion. A derivation of the system from a mass transport viewpoint is given in [7]. For an analysis of the steady state solutions and for further references we refer to [6], [12], [13], and [20].

In this paper we study the corresponding system in higher space dimension, namely

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \operatorname{div}(\sigma_i \nabla u_i + u_i \nabla \psi_i) \\ &\quad + \alpha_i \left(\sum_{j=1}^n \lambda_{ij} r_j(u_j(x, t), x) \right) \quad \text{in } Q_T, \end{aligned} \tag{1.3a}$$

where $i \in \{1, \dots, n\}$, and $u_i(x, t) : Q_T \rightarrow \mathbb{R}^+$, with $Q_T = \Omega \times (0, T)$, Ω an open bounded subset of \mathbb{R}^d with smooth boundary, and T some positive constant. We supplement this system with the Robin (no-flux) boundary conditions

$$\sigma_i \frac{\partial u_i}{\partial \nu} + u_i \frac{\partial \psi_i}{\partial \nu} = 0, \quad i \in \{1, \dots, n\}, \quad \text{on } \partial \Omega \times (0, T), \tag{1.3b}$$

where ν is the outward normal vector to $\partial\Omega$, and the initial conditions

$$u_1(x, 0) = u_{0,1}(x), \dots, u_n(x, 0) = u_{0,n}(x), \quad x \in \Omega. \quad (1.3c)$$

We assume that the following hypotheses hold

1. The constants σ_i and $\alpha_i \in \mathbb{R}$, where $i \in \{1, \dots, n\}$, are strictly positive;
2. For $i, j \in \{1, \dots, n\}$, $\lambda_{ii} \leq 0$, $\lambda_{ij} \geq 0$ if $i \neq j$, $\sum_{k=1}^n \lambda_{kj} = 0$;
3. for all $i \in \{1, \dots, n\}$, the smooth functions r_i are nondecreasing with respect to the first variable; $r_i(0, x) = 0$ and we assume that the functions ψ_i are smooth as well;
4. $u_i(\cdot, 0) = u_{0i} \in C(\overline{\Omega})$, $u_{0i} \geq 0$.

In the linear case of the molecular motors, it amounts to choosing

$$r_i(s, x) = s, \quad \lambda_{ij} = a_{ij} \text{ and } \alpha_i = 1 \text{ for all } i, j \in \{1, \dots, n\}. \quad (1.4)$$

We denote by Problem (P) the system (1.3a) together with the boundary and initial conditions (1.3b), (1.3c), and admit without proof that Problem (P) possesses a unique smooth and bounded solution on each time interval $(0, T]$. An essential idea for proving the existence of a solution would be to apply the Comparison principle Theorem 2.2 below to deduce that any solution of Problem (P) has to be nonnegative and bounded from above by a stationary solution.

Finally, we note that because of the boundary conditions (1.3b) the quantity

$$\sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} u_i(x, t) \, dx \quad (1.5)$$

is conserved in time.

The organization of this paper is as follows. In Section 2 we prove a comparison principle for Problem (P). The main idea, which permits to show that Problem (P) is cooperative, is a change of functions which transforms the Robin boundary conditions into the homogeneous Neumann boundary conditions. In Section 3 we establish a contraction in L^1 property for the corresponding semigroup solution. Let us point out the similarity with an old result due to Crandall and Tartar [8] where they proved in a scalar case that in the presence of a conservation of the integral property such as (1.5), a comparison principle such as Theorem 2.2 is equivalent to a contraction in L^1 property such as the inequality (3.4) below. As far as we know such an abstract result is not known in the case of systems.

Section 4 deals with the large time behavior of the solutions. Supposing the existence of a stationary solution, we construct a continuum of

stationary solutions and prove that the solutions stabilize as t tends to infinity. Let us mention a result by Perthame [19] who proved the stabilization in the case of the two component one-dimensional molecular motor problem. Finally in Section 5, show the existence and uniqueness (up to a multiplicative constant) of the stationary solution of the molecular motor problem.

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2 Comparison principle

First, we remark that the system of equations (1.3a) is cooperative. However, since nothing is known about the sign of the coefficients $\frac{\partial \psi_i}{\partial \nu}$ in the Robin boundary conditions (1.3b), we cannot decide whether the Problem (P) is cooperative. This leads us to perform a change of variables which transforms the Robin boundary conditions into the homogeneous Neumann boundary conditions.

2.1 The change of unknown functions

Performing the change of variables

$$w_i(x, t) = u_i(x, t) e^{\psi_i(x)/\sigma_i}, \quad i \in \{1, \dots, n\}, \quad (2.1)$$

we deduce from (1.3) that $\vec{w} := (w_1, \dots, w_n)$ satisfies the parabolic problem

$$\begin{aligned} \frac{\partial w_i}{\partial t} &= \sigma_i e^{\psi_i(x)/\sigma_i} \operatorname{div} \left(e^{-\psi_i(x)/\sigma_i} \nabla w_i \right) \\ &+ \alpha_i e^{\psi_i(x)/\sigma_i} \left(\sum_{j=1}^n \lambda_{ij} r_j (w_j(x, t) e^{-\psi_j(x)/\sigma_j}, x) \right) \quad \text{in } Q_T, \end{aligned} \quad (2.2)$$

together with the homogeneous Neumann boundary conditions

$$\frac{\partial w_i}{\partial \nu} = 0, \quad i \in \{1, \dots, n\}, \quad \text{on } \partial\Omega, \quad (2.3)$$

and the initial conditions

$$w_i(x, 0) = u_{0,i}(x) e^{\psi_i(x)/\sigma_i}, \quad i \in \{1, \dots, n\}, \quad x \in \Omega. \quad (2.4)$$

In the following, we denote by Problem P_N — the problem (2.2), (2.3), (2.4). To begin with we define the operators

$$\begin{aligned} \mathcal{L}_i(w_i) &= \frac{\partial w_i}{\partial t} - \sigma_i e^{\psi_i(x)/\sigma_i} \operatorname{div} \left(e^{-\psi_i(x)/\sigma_i} \nabla w_i \right) \\ &\quad - \alpha_i e^{\psi_i(x)/\sigma_i} \left(\sum_{j=1}^n \lambda_{ij} r_j(w_j(x, t) e^{-\psi_j(x)/\sigma_j}, x) \right) \quad \text{in } Q_T. \end{aligned} \quad (2.5)$$

We say that $(\underline{w}_1, \dots, \underline{w}_n)$ is a subsolution of Problem P_N if

$$\begin{aligned} \mathcal{L}_i(\underline{w}_i) &\leq 0 \quad \text{in } Q_T, \\ \frac{\partial \underline{w}_i}{\partial \nu} &\leq 0 \quad \text{on } \partial\Omega \times (0, T), \\ \underline{w}_i(x, 0) &\leq w_i(x, 0), \quad x \in \Omega \end{aligned} \quad (2.6)$$

for all $i \in \{1, \dots, n\}$. We define similarly a supersolution $(\overline{w}_1, \dots, \overline{w}_n)$ of Problem P_N by the inequalities

$$\begin{aligned} \mathcal{L}_i(\overline{w}_i) &\geq 0 \quad \text{in } Q_T, \\ \frac{\partial \overline{w}_i}{\partial \nu} &\geq 0 \quad \text{on } \partial\Omega \times (0, T), \\ \overline{w}_i(x, 0) &\geq w_i(x, 0), \quad x \in \Omega. \end{aligned} \quad (2.7)$$

The following comparison theorem holds ([2], [21]).

Theorem 2.1. *Let $(\underline{w}_1, \dots, \underline{w}_n)$ and $(\overline{w}_1, \dots, \overline{w}_n)$, be a sub- and a super-solution, respectively, for the operators \mathcal{L}_j defined by (2.5) with $j \in \{1, \dots, n\}$, which means that (2.6) and (2.7) hold for $i \in \{1, \dots, n\}$. Then $\underline{w}_i \leq \overline{w}_i$ in Q_T . Moreover, for all $i \in \{1, \dots, n\}$ such that $\underline{w}_i \leq \overline{w}_i$ and $\underline{w}_i \not\equiv \overline{w}_i$ on $\{t = 0\} \times \Omega$ then $\underline{w}_i < \overline{w}_i$ in Q_T . ■*

This comparison theorem immediately translates into a comparison theorem for solutions of the original Problem (P). For all $i \in \{1, \dots, n\}$, we define the operators

$$\begin{aligned} L_i(u_i) &= (u_i)_t - \operatorname{div}(\sigma_i \nabla u_i + u_i \nabla \psi_i) \\ &\quad - \alpha_i \left(\sum_{j=1}^n \lambda_{ij} r_j(u_j, x) \right) \quad \text{in } Q_T. \end{aligned} \quad (2.8)$$

The following result holds.

Theorem 2.2. *Let $(\underline{u}_1, \dots, \underline{u}_n)$ and $(\overline{u}_1, \dots, \overline{u}_n)$, be a sub- and a super-solution, respectively, for the operators L_j , defined by (2.8) with $j \in \{1, \dots, n\}$. Then $\underline{u}_i \leq \overline{u}_i$ in Q_T . Moreover, for all $i \in \{1, \dots, n\}$ such that $\underline{u}_i \leq \overline{u}_i$ and $\underline{u}_i \not\equiv \overline{u}_i$ on $\{t = 0\} \times \Omega$ then $\underline{u}_i < \overline{u}_i$ in Q_T . ■*

Next we state two immediate corollaries of Theorem 2.2.

Corollary 2.3. (*uniqueness*) If (u_1^1, \dots, u_n^1) and (u_1^2, \dots, u_n^2) are solutions of Problem (P) with the same initial condition $(u_{0,1}, \dots, u_{0,n}) \in (C(\overline{\Omega}))^n$, then for all $i \in \{1, \dots, n\}$, $u_i^1 = u_i^2$. ■

Corollary 2.4. (*positivity*) If (u_1, \dots, u_n) is the solution of Problem (P) with the nonnegative initial condition $(u_{0,1}, \dots, u_{0,n}) \in (C(\overline{\Omega}))^n$, then for all $i \in \{1, \dots, n\}$, $u_i \geq 0$. Moreover, for all $i \in \{1, \dots, n\}$, such that $u_{0,i} \geq 0$ and $u_{0,i} \neq 0$, $u_i > 0$ in Ω . ■

3 Contraction property

The purpose of this section is to show a contraction in $(L^1(\Omega))^n$ property for the solutions of Problem (P) with the initial conditions belonging to $(L^\infty(\Omega))^n$. The main steps of the proof rely upon arguments due to [3] and [18].

We first introduce some notation. We suppose that the functions (u_1^1, \dots, u_n^1) and (u_1^2, \dots, u_n^2) are the solutions of Problem (P) with the initial conditions $(u_{0,1}^1, \dots, u_{0,n}^1)$ and $(u_{0,1}^2, \dots, u_{0,n}^2)$, respectively. Define

$$(U_1, \dots, U_n) := (u_1^1 - u_1^2, \dots, u_n^1 - u_n^2). \quad (3.1)$$

Then

$$\begin{aligned} (U_i)_t &= \operatorname{div}(\sigma_i \nabla U_i + U_i \nabla \psi_i) \\ &+ \alpha_i \sum_{j=1}^n \lambda_{ij} (r_j(u_j^1(x, t), x) - r_j(u_j^2(x, t), x)) \quad \text{in } Q_T, \\ \sigma_i \frac{\partial U_i}{\partial \nu} + U_i \frac{\partial \psi_i}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ U_i(x, 0) &= U_{0,i}(x) \quad \text{for } x \in \Omega, \end{aligned} \quad (3.2)$$

together with

$$U_{0,i} = u_{0,i}^1 - u_{0,i}^2, \quad (3.3)$$

for each $i \in \{1, \dots, n\}$.

Next we prove the following contraction in L^1 property.

Theorem 3.1. For all $t > 0$,

$$\begin{aligned} \frac{1}{\alpha_1} \|U_1(\cdot, t)\|_{L^1(\Omega)} + \dots + \frac{1}{\alpha_n} \|U_n(\cdot, t)\|_{L^1(\Omega)} \\ \leq \frac{1}{\alpha_1} \|U_{0,1}(\cdot)\|_{L^1(\Omega)} + \dots + \frac{1}{\alpha_n} \|U_{0,n}(\cdot)\|_{L^1(\Omega)}, \end{aligned} \quad (3.4)$$

where U_i and $U_{0,i}$, $i \in \{1, \dots, n\}$, are defined by (3.1) and (3.3), respectively.

Proof Dividing each partial differential equation of (3.2) by α_i and summing them up, we obtain

$$\begin{aligned}
\frac{d}{dt} \left(\sum_{i=1}^n \frac{1}{\alpha_i} U_i \right) &= \sum_{i=1}^n \frac{1}{\alpha_i} \operatorname{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i) \\
&\quad + \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} \left(r_j(u_j^1(x, t), x) - r_j(u_j^2(x, t), x) \right) \\
&= \sum_{i=1}^n \frac{1}{\alpha_i} \operatorname{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i) \\
&\quad + \sum_{j=1}^n \left\{ \left(r_j(u_j^1(x, t), x) - r_j(u_j^2(x, t), x) \right) \sum_{i=1}^n \lambda_{ij} \right\} \\
&= \sum_{i=1}^n \frac{1}{\alpha_i} \operatorname{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i),
\end{aligned}$$

where we have used Hypothesis 2.

This, together with the boundary conditions (1.3b), implies the conservation in time of the quantity

$$\frac{d}{dt} \sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} U_i(x, t) dx = 0. \quad (3.5)$$

Let us look closer at the nonlinear term in (3.2). We can write, for fixed index i

$$\begin{aligned}
&\sum_{j=1}^n \lambda_{ij} (r_j(u_j^1(x, t), x) - r_j(u_j^2(x, t), x)) \\
&= \sum_{j=1}^n \lambda_{ij} U_j \int_0^1 \frac{\partial}{\partial u} r_j(\theta u_j^1 + (1 - \theta) u_j^2, x) d\theta = \sum_{j=1}^n A_{ij} U_j.
\end{aligned}$$

Freezing the functions u_i^k for $i \in \{1, \dots, n\}$, $k \in \{1, 2\}$, we deduce that the functions U_1, \dots, U_n satisfy a system of the form

$$(U_i)_t = \operatorname{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i) + \sum_{j=1}^n A_{ij} U_j \quad \text{in } Q_T, \quad (3.6)$$

with the boundary and initial conditions

$$\begin{aligned}
\sigma_i \frac{\partial U_i}{\partial \nu} + U_i \frac{\partial \psi_i}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
U_i(x, 0) &= U_{0,i}(x), \quad x \in \Omega.
\end{aligned} \quad (3.7)$$

for $i \in \{1, \dots, n\}$, where A_{ij} are functions of space and time.

In order to make the notation more concise, we write

$$\begin{aligned}\vec{U}_0 &= (U_{0,1}, \dots, U_{0,n}), \\ \vec{U} &= (U_1, \dots, U_n), \\ \vec{U}_0^\pm &= (U_{0,1}^\pm, \dots, U_{0,n}^\pm), \\ \vec{U}^\pm &= (U_1^\pm, \dots, U_n^\pm),\end{aligned}$$

where $s^+ = \max\{s, 0\}$, $s^- = \max\{-s, 0\}$. By (3.6), (3.7) and Corollary 2.3 we can write \vec{U} in the form

$$(\vec{U})(x, t) = \mathcal{S}(t) \vec{U}_0(x) = (\mathcal{S}_1(t) \vec{U}_0, \dots, \mathcal{S}_n(t) \vec{U}_0)(x)$$

with some operator $\mathcal{S}(t)$. We set

$$(W_1, \dots, W_n) = -(U_1 e^{\psi_1(x)/\sigma_1}, \dots, U_n e^{\psi_n(x)/\sigma_n}),$$

and $\tilde{A}_{ij} = A_{ij} e^{\psi_i(x)/\sigma_i} e^{-\psi_j(x)/\sigma_j}$. Then, the system of equations (3.6) can be expressed in the form

$$(W_i)_t = \sigma_i e^{\psi_i(x)/\sigma_i} \operatorname{div} \left(e^{-\psi_i(x)/\sigma_i} \nabla W_i \right) + \sum_{j=1}^n \tilde{A}_{ij} W_j \quad \text{in } Q_T, \quad (3.8)$$

with the boundary and initial conditions

$$\frac{\partial W_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3.9)$$

$$W_i(x, 0) = -U_{0,i} e^{\psi_i(x)/\sigma_i}, \quad x \in \Omega, \quad (3.10)$$

for $i \in \{1, \dots, n\}$.

Next we show that the solutions W_i of the problem (3.8)–(3.10) with nonpositive initial conditions are nonpositive in $\overline{\Omega}$ for all $t \in (0, T)$. To that purpose we consider the auxiliary problem

$$(W_i)_t - \vartheta_i(x) \operatorname{div} \left(\zeta_i(x) \nabla W_i \right) - \sum_{j=1}^n \gamma_{ij} W_j \leq 0 \quad \text{in } Q_T, \quad (3.11)$$

$$\frac{\partial W_i}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3.12)$$

$$W_i(x, 0) = W_{0,i}(x) \leq 0 \quad x \in \Omega, \quad (3.13)$$

for $i \in \{1, \dots, n\}$. We assume that $\vartheta_i(x)$ and $\zeta_i(x)$ are nonnegative in $\overline{\Omega}$ and that the coefficients γ_{ij} satisfy the same assumptions as the coefficients λ_{ij} in Problem (P). The following result holds.

Lemma 3.2. *Let (W_1, \dots, W_n) be a smooth and bounded solution of the problem (3.11) – (3.13) with nonpositive initial conditions $W_{0,i}$ on a time interval $[0, T]$. Then $W_i(x, t) \leq 0$ in $\overline{\Omega} \times (0, T]$. Moreover, for each $i \in \{1, \dots, n\}$ such that $W_{0,i} \leq 0$ and $W_{0,i} \not\equiv 0$, $W_i < 0$ in $\overline{\Omega} \times (0, T]$.*

Proof The result of Lemma 3.2 follows from the fact that the system (3.11), (3.12), (3.13), with the inequalities $\{\leq\}$ replaced by the equalities $\{=\}$, is a cooperative system. However, for the sake of completeness, we present a proof below. We first remark that, in view of [21, Remark (i), p. 191], one can always satisfy the condition

$$\sum_{j=1}^n \gamma_{ij} \leq 0 \text{ for all } i \in \{1, \dots, n\}, \quad (3.14)$$

for the matrix of coefficients $(\gamma_{ij})_{i,j=1}^n$ by performing the change of variables $\overline{W}_i = W_i e^{-ct}$ for all $i \in \{1, \dots, n\}$ and $c > 0$ large enough. Thanks to the regularity of each W_i , we can apply Theorem 15, p. 191 from [21] to conclude that $W_i - M \leq 0$ in $\overline{\Omega} \times [0, T]$ for some $M > 0$ and all $i \in \{1, \dots, n\}$. In fact, we can deduce that $W_i - M < 0$ in $\overline{\Omega} \times (0, T)$.

Indeed, if for some $k \in \{1, \dots, n\}$, $W_k = M$ in an interior point $(\tilde{x}, \tilde{t}) \in \Omega \times (0, T)$, then Theorem 15, p. 191 in [21] implies that $W_k \equiv M$ for all $0 \leq t < \tilde{t}$, which is impossible since $W_k(x, 0) \leq 0$. If the maximum M of W_k is attained at a boundary point $P \in \partial\Omega \times (0, T)$ then either there exists an open ball $K \subset \Omega \times (0, T)$ such that $P \in \partial K$ and $W_k - M < 0$ in K , and the last part of Theorem 15, p. 191 in [21] contradicts the boundary inequality (3.12), or for all open balls $K \subset \Omega \times (0, T)$ such that $P \in \partial K$ there exists a point $(\tilde{x}, \tilde{t}) \in K$ such that $W_i(\tilde{x}, \tilde{t}) = M$, and we proceed as in the case before.

Hence, there exists $\widetilde{M} > 0$, such that $W_i \leq \widetilde{M} < M$ in $\overline{\Omega} \times [0, T]$ for all $i \in \{1, \dots, n\}$. Then we can repeat the reasoning for all $M > 0$ until $M = 0$. Indeed, if this would not be the case, we find the least real number $\overline{M} > 0$, with $W_i \leq \overline{M} \leq \widetilde{M}$ in $\overline{\Omega} \times [0, T]$, which leads again to the existence of a real number $0 \leq \widehat{M} < \overline{M}$ with the same property. This contradicts the fact that \overline{M} was defined as the least such real number. ■

Since the functions u_i^1 , u_i^2 are bounded on $\overline{\Omega} \times [0, T]$, it follows that the functions W_i are bounded on $\overline{\Omega} \times [0, T]$ for all $i \in \{1, \dots, n\}$. Then we are in a position to apply Lemma 3.2 with $\vartheta_i(x) = e^{\psi_i/\sigma_i}$, $\zeta_i(x) = \sigma_i e^{-\psi_i/\sigma_i}$ and $\gamma_{ij} = \widetilde{A}_{ij}$ for $i, j \in \{1, \dots, n\}$. We deduce that the solutions W_i of the problem (3.8) – (3.10) with nonpositive initial conditions are nonpositive in $\overline{\Omega}$ for all $t \in (0, T)$.

Next we remark that the above reasoning can be applied either with \vec{U}_0 replaced by U_0^+ or with \vec{U}_0 replaced by U_0^- . This permits to show that $\mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \geq 0$ and that

$$\mathcal{S}_i(t)\vec{U}_0^\pm > 0 \quad \text{if} \quad \vec{U}_0^\pm \neq 0. \quad (3.15)$$

We easily compute

$$\begin{aligned} & \sum_{i=1}^n \frac{1}{\alpha_i} \|U_i(\cdot, t)\|_{L^1(\Omega)} - \sum_{i=1}^n \frac{1}{\alpha_i} \|U_{0,i}(\cdot)\|_{L^1(\Omega)} \\ &= \sum_{i=1}^n \frac{1}{\alpha_i} \|\mathcal{S}_i(t)\vec{U}_0^+ - \mathcal{S}_i(t)\vec{U}_0^-\|_{L^1(\Omega)} - \sum_{i=1}^n \frac{1}{\alpha_i} \|U_{0,i}(\cdot)\|_{L^1(\Omega)} \\ &= \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} \left\{ \max \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} \right. \\ & \quad \left. - \frac{1}{\alpha_i} \min \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} \right\} dx - \sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} \{U_{i,0}^+ + U_{i,0}^-\} dx \\ &= \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} (\mathcal{S}_i(t)\vec{U}_0^+ + \mathcal{S}_i(t)\vec{U}_0^-) dx - \sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} \{U_{i,0}^+ + U_{i,0}^-\} dx \\ & \quad - 2 \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} \min \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} dx \\ &= -2 \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} \min \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} dx \leq 0, \end{aligned} \quad (3.16)$$

$$\begin{aligned} & - \frac{1}{\alpha_i} \min \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} \} dx - \sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} \{U_{i,0}^+ + U_{i,0}^-\} dx \\ &= \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} (\mathcal{S}_i(t)\vec{U}_0^+ + \mathcal{S}_i(t)\vec{U}_0^-) dx - \sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} \{U_{i,0}^+ + U_{i,0}^-\} dx \\ & \quad - 2 \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} \min \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} dx \\ &= -2 \sum_{i=1}^n \int_{\Omega} \frac{1}{\alpha_i} \min \{ \mathcal{S}_i(t)\vec{U}_0^+, \mathcal{S}_i(t)\vec{U}_0^- \} dx \leq 0, \end{aligned} \quad (3.17)$$

which completes the proof of (3.4). ■

Corollary 3.3. *Let $(u_{0,1}^1, \dots, u_{0,n}^1), (u_{0,1}^2, \dots, u_{0,n}^2) \in (C(\overline{\Omega}))^n$ be as in Theorem 3.1. Moreover, let us assume that for at least one index $k \in \{1, \dots, n\}$ the difference $u_{0,k}^1 - u_{0,k}^2$ changes the sign. Then, the inequality (3.4) is strict for all $t > 0$, so that solution satisfies a strict contraction property.*

4 Large time behavior of solutions

In this section we assume the existence and uniqueness of a positive solution $\vec{v} = (v_1, \dots, v_n) \in (C(\overline{\Omega}) \cap C^2(\Omega))^n$ of the elliptic problem

$$\operatorname{div}(\sigma_i \nabla v_i + v_i \nabla \psi_i) + \alpha_i \left(\sum_{j=1}^n \lambda_{ij} r_j(v_j(x), x) \right) = 0 \quad \text{in } \Omega, \quad (4.1)$$

$$\sigma_i \frac{\partial v_i}{\partial \nu} + v_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (4.2)$$

$$\sum_{i=1}^n \frac{1}{\alpha_i} \int_{\Omega} v_i(x) \, dx = 1, \quad (4.3)$$

for $i \in \{1, \dots, n\}$.

Definition 4.1. *We say that a vector function $\vec{v} = (v_1, \dots, v_n) \in (C(\overline{\Omega}))^n$ is nonnegative (resp. positive) if $v_i(x) \geq 0$ (resp. $v_i(x) > 0$) for all $x \in \overline{\Omega}$ and all $i \in \{1, \dots, n\}$.*

Next we introduce the semigroup notation for the unique solution of Problem (P), namely

$$\vec{u}(t) = \mathcal{T}(t) \vec{u}_0 = (\mathcal{T}_1(t) \vec{u}_0, \dots, \mathcal{T}_n(t) \vec{u}_0),$$

with the initial data $\vec{u}_0 \in (C(\overline{\Omega}))^n$. The method of the proof is based upon an idea of Osher and Ralston [18]. It mainly exploits the contraction properties for the nonlinear semigroup $\mathcal{T}(t)$ given by Theorem 3.1 and Corollary 3.3. A similar reasoning was developed in other contexts by Bertsch and Hilhorst [3], Hilhorst and Hulshof [14] and Hilhorst and Peletier [15].

We suppose there exists a set $\mathcal{H} \subset (C(\overline{\Omega}) \cap C^2(\Omega))^n$ of positive stationary solutions with the following property which we denote by \mathcal{S} :

For each $\vec{f} = (f_1, \dots, f_n) \in (C(\overline{\Omega}) \cap C^2(\Omega))^n$ either $\vec{f} \in \mathcal{H}$ or there exists $(\xi_1, \dots, \xi_n) \in \mathcal{H}$, such that $f_i - \xi_i$ changes the sign for at least one index $i \in \{1, \dots, n\}$.

One can prove that a set \mathcal{H} satisfying Property \mathcal{S} exists in at least two cases:

- i)* In the case of the system (1.1) where the Robin boundary conditions reduce to the homogeneous Neumann boundary conditions,

the set \mathcal{H} is given by

$$\mathcal{H} = \left\{ (a, b) : a > 0, b = r_B^{-1}(r_A(a)) \right. \\ \left. \text{and } \frac{a}{\alpha} + \frac{b}{\beta} = \int_{\Omega} \left(\frac{u}{\alpha} + \frac{v}{\beta} \right) dx \right\}.$$

For more details we refer to [5].

- ii) In the case of the molecular motor with a linear n -component system the set \mathcal{H} is given by

$$\mathcal{H} = \{ c\vec{v} : c \in \mathbb{R}^+ \},$$

where \vec{v} is a unique solution of the elliptic problem (4.1)–(4.3).

Proposition 4.2. *The continuum \mathcal{H} is such that for each*

$$\vec{f} = (f_1, \dots, f_n) \in (C(\overline{\Omega}) \cap C^2(\Omega))^n$$

either $\vec{f} \in \mathcal{H}$, or there exists $(\xi_1, \dots, \xi_n) \in \mathcal{H}$ such that $f_i - \xi_i$ changes the sign for at least one index $i \in \{1, \dots, n\}$.

Proof

- i) In the case of system (1.1) the proof is rather obvious since the continuum \mathcal{H} is composed of constant pairs.
- ii) In the case of the molecular motor, let us assume that $\vec{f} \notin \mathcal{H}$. Then there does not exist any positive constant c such that $c\vec{v} = \vec{f}$. In particular, there exists an index $i \in \{1, \dots, n\}$ such that v_i is not proportional to f_i , or in other words $cv_i \neq f_i$ for all $c > 0$. Without loss of generality we can assume that the first coordinate has this property. Let $x_0 \in \Omega$ be arbitrary. Since v_1 is strictly positive in $\overline{\Omega}$, we can define

$$c_0 = \frac{f_1(x_0)}{v_1(x_0)},$$

so that

$$(f_1 - c_0 v_1)(x_0) = 0.$$

Let $\mathcal{Z} = \{x \in \overline{\Omega} : (f_1 - c_0 v_1)(x) = 0\}$. From the continuity of f_1 and v_1 , \mathcal{Z} is closed as a subset of Ω . If there exist $x_1, x_2 \in \mathcal{Z}^c$, such that $(f_1 - c_0 v_1)(x_1)$ and $(f_1 - c_0 v_1)(x_2)$ are of different signs, then the proof is complete. Now suppose that $(f_1 - c_0 v_1)(x)$ is positive for all $x \in \mathcal{Z}^c$. In particular

$$(f_1 - c_0 v_1)(\tilde{x}) = d > 0$$

for some fixed $\tilde{x} \in \mathcal{Z}^c$. Then choosing $\varepsilon = \frac{d}{2v_1(\tilde{x})}$ we see that

$$(f_1 - (c_0 + \varepsilon)v_1)(\tilde{x}) = \frac{d}{2} > 0.$$

However

$$(f_1 - (c_0 + \varepsilon)v_1)(x_0) < 0.$$

We proceed similarly when $(f_1 - c_0 v_1)(x)$ is negative for all $x \in \mathcal{Z}^c$. ■

In the sequel we suppose that the initial data $\vec{u}_0 = (u_{0,1}, \dots, u_{0,n})$ from $(C(\overline{\Omega}))^n$ also satisfy the following property:

$$\text{There exists } \vec{h} \in \mathcal{H} \text{ such that } 0 \leq \vec{u}_0 \leq \vec{h} \text{ in } \overline{\Omega}, \quad (4.4)$$

and remark that this property is satisfied in both the cases (i) and (ii).

Proposition 4.3. *Let $\vec{u}_0 = (u_{0,1}, \dots, u_{0,n}) \in (C(\overline{\Omega}))^n$ satisfy the property (4.4). Then the solution (u_1, \dots, u_n) of Problem (P) is such that $0 \leq \vec{u}(t) \leq \vec{h}$ for all $t > 0$.*

Proof We remark that $\vec{0}$ is a subsolution of Problem (P) and that \vec{h} is a supersolution, and apply Theorem 2.2. ■

Next we prove the main result of this section. To that purpose we first define the norm $\|\cdot\|_1$ by

$$\|\vec{f}\|_1 = \sum_{i=1}^n \frac{1}{\alpha_i} \|f_i\|_{L^1(\Omega)}.$$

Note that this norm is equivalent to the usual product norm in the space $(L^1(\Omega))^n$.

Theorem 4.4. *For all nonnegative $\vec{u}_0 = (u_{0,1}, \dots, u_{0,n}) \in (C(\overline{\Omega}))^n$ there exists $\vec{f} = (f_1, \dots, f_n) \in \mathcal{H}$, such that*

$$\lim_{t \rightarrow \infty} \|\mathcal{T}(t) \vec{u} - \vec{f}\|_1 = 0.$$

Proof

The proof consists of several steps. To begin with we define the ω -limit set

$$\begin{aligned} \omega(\vec{u}_0) = \Big\{ \vec{g} \in (L^1(\Omega))^n : \text{there exists a sequence } t_k \rightarrow \infty \\ \text{as } k \rightarrow \infty, \text{ such that } \lim_{k \rightarrow \infty} \|\mathcal{T}(t_k) \vec{u}_0 - \vec{g}\|_1 = 0 \Big\}, \end{aligned} \quad (4.5)$$

The organization of the proof is as follows. First we show that $\omega(\vec{u}_0)$ is not empty. In the second step we define the Lyapunov functional

$$\mathcal{V}(\vec{\xi}) = \|\vec{\xi} - \vec{w}\|_1,$$

where \vec{w} is a stationary solution and check that it is constant on $\omega(\vec{u}_0)$. We then deduce that $\omega(\vec{u}_0) \subset \mathcal{H}$, and finally prove that $\omega(\vec{u}_0)$ consists of exactly one function.

Step 1. $\omega(\vec{u}_0)$ is not empty.

Let $\varepsilon > 0$ be arbitrary. Suppose that $\Omega' \subset \subset \Omega$ satisfy

$$|\Omega \setminus \Omega'| \leq \frac{\varepsilon}{2K}.$$

and set

$$K = \sum_{i=1}^n \frac{2}{\alpha_i} \|h_i\|_{C(\overline{\Omega})}, \quad (4.6)$$

where \vec{h} has been introduced in (4.4). We have already proved in Proposition 4.3 that $\mathcal{T}(t)\vec{u}_0$ is bounded in $(L^\infty(\Omega))^n$. Therefore there exist a vector function $\vec{g} \in (L^\infty(\Omega))^n$ and a sequence $\{\vec{u}(t_k)\}$ such that

$$\vec{u}(t_k) \rightharpoonup \vec{g} \text{ weakly in } (L^2(\Omega))^n, \quad (4.7)$$

as $t_k \rightarrow \infty$. Next we deduce from [16, Chap. III, Theorem 10.1] that there exists a positive constant C such that

$$|u_i(x_1, t) - u_i(x_2, t)| \leq C|x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in \Omega'$ and all $t > 0$. Therefore, it follows from the Ascoli-Arzelà Theorem (see, e.g., [1, Theorem 1.33]) that $\vec{u}(t_k) \rightarrow \vec{g}$ as $t_k \rightarrow \infty$, uniformly in $\overline{\Omega}'$. We choose t_0 large enough such that for all $t_k \geq t_0$

$$\|\vec{u}(\cdot, t_k) - \vec{g}(\cdot)\|_{1, \Omega'} \leq \frac{\varepsilon}{2}, \quad (4.8)$$

where $\|\cdot\|_{1, \Omega'}$ corresponds to the L^1 norm in Ω' . We deduce that, in view of (4.6) and (4.7) that

$$\|\vec{u}(\cdot, t_k) - \vec{g}(\cdot)\|_{1, \Omega \setminus \Omega'} \leq K|\Omega \setminus \Omega'| \leq \frac{\varepsilon}{2},$$

which together with (4.8) yields

$$\|\vec{u}(\cdot, t_k) - \vec{g}(\cdot)\|_1 \leq \varepsilon.$$

Step 2. $\omega(\vec{u}_0) \subset \mathcal{H}$.

Indeed, let $\vec{g} \in \omega(\vec{u}_0)$ and suppose $\vec{g} \notin \mathcal{H}$. According to Proposition

4.2 we can find a steady state solution $\vec{w} \in \mathcal{H}$, such that at least one component of $\vec{w} - \vec{g}$ changes the sign. Without loss of generality we can assume that it happens for the first component, namely that $f_1 - w_1$ changes the sign. We remark that, by the contraction property in Theorem 3.1, the functional

$$\mathcal{V}(\vec{\xi}) = \|\vec{\xi} - \vec{w}\|_1$$

is a Lyapunov functional for Problem (P), where $\vec{\xi} \in (L^1(\Omega))^n$. Next we describe some of its properties.

Property (a) *The functional \mathcal{V} is constant on $\omega(\vec{u}_0)$.*

Since $\mathcal{T}(t)\vec{w} = \vec{w}$ and $\mathcal{T}(t)$ has the contraction property (3.4), the functional \mathcal{V} is nonincreasing in time along the trajectory $\mathcal{T}(t)\vec{u}_0$, which yields

$$\begin{aligned} \mathcal{V}(\mathcal{T}(t)\vec{u}_0) &= \|\mathcal{T}(t)\vec{u}_0 - \vec{w}\|_1 \\ &= \|\mathcal{T}(t)\vec{u}_0 - \mathcal{T}(t)\vec{w}\|_1 \leq \|\vec{u}_0 - \vec{w}\|_1 < \infty. \end{aligned}$$

Thus there exists a finite limit \mathcal{V}^* of $\mathcal{V}(\mathcal{T}(t)\vec{u}_0)$ as $t \rightarrow \infty$. Let $\vec{h}_1, \vec{h}_2 \in \omega(\vec{u}_0)$. We can find a sequence $t_k \rightarrow \infty$ as $k \rightarrow \infty$, such that

$$\|\mathcal{T}(t_{2k})\vec{u}_0 - \vec{h}_1\|_1 \rightarrow 0 \quad \text{and} \quad \|\mathcal{T}(t_{2k+1})\vec{u}_0 - \vec{h}_2\|_1 \rightarrow 0,$$

as k tends to ∞ . It follows that $\mathcal{V}(\vec{h}_1) = \mathcal{V}(\vec{h}_2) = \mathcal{V}^*$.

Property (b) *The ω -limit set $\omega(\vec{u}_0)$ is invariant with respect to the semigroup $\mathcal{T}(t)$, namely if $\vec{h} \in \omega(\vec{u}_0)$, then for all $t > 0$ also $\mathcal{T}(t)\vec{h} \in \omega(\vec{u}_0)$.*

Let the sequence $t_k \rightarrow \infty$ as $k \rightarrow \infty$ be such that $\|\mathcal{T}(t_k)\vec{u}_0 - \vec{h}\|_1 \rightarrow 0$. From the contraction property (3.4)

$$\begin{aligned} \|\mathcal{T}(t_k + t)\vec{u}_0 - \mathcal{T}(t)\vec{h}\|_1 &= \|\mathcal{T}(t)\mathcal{T}(t_k)\vec{u}_0 - \mathcal{T}(t)\vec{h}\|_1 \\ &\leq \|\mathcal{T}(t_k)\vec{u}_0 - \vec{h}\|_1. \end{aligned}$$

Since the last term above tends to 0 as k tends to ∞ this shows that $\mathcal{T}(t)\vec{h} \in \omega(\vec{u}_0)$.

Now, remember that $\vec{g} \in \omega(\vec{u}_0)$ is such that $\vec{g} \notin \mathcal{H}$ and $\vec{w} \in \mathcal{H}$ is such that the first component of $\vec{w} - \vec{g}$ changes the sign in Ω . Then, Corollary 3.3 yields

$$\begin{aligned} \mathcal{V}(\mathcal{T}(t)\vec{g}) &= \|\mathcal{T}(t)\vec{g} - \vec{w}\|_1 \\ &= \|\mathcal{T}(t)\vec{g} - \mathcal{T}(t)\vec{w}\|_1 < \|\vec{g} - \vec{w}\|_1 = \mathcal{V}(\vec{g}), \end{aligned}$$

for all $t > 0$, which contradicts Property (a). Therefore $\vec{g} \in \mathcal{H}$.

Step 3. *The set $\omega(\vec{u}_0)$ contains only one element.*

Suppose that $\vec{g}_1, \vec{g}_2 \in \omega(\vec{u}_0)$. Then we can find two sequences t_k, s_k tending to ∞ as $k \rightarrow \infty$, such that $s_k \leq t_k$ and $\|\mathcal{T}(t_k) \vec{u}_0 - \vec{g}_1\|_1, \|\mathcal{T}(s_k) \vec{u}_0 - \vec{g}_2\|_1 \rightarrow 0$ as $t_k \rightarrow \infty$. Since $\omega(\vec{u}_0) \subset \mathcal{H}$, it follows that

$$\begin{aligned} \|\vec{g}_1 - \vec{g}_2\|_1 &\leq \|\mathcal{T}(t_k) \vec{u}_0 - \vec{g}_1\|_1 + \|\mathcal{T}(t_k) \vec{u}_0 - \vec{g}_2\|_1 \\ &= \|\mathcal{T}(t_k) \vec{u}_0 - \vec{g}_1\|_1 + \|\mathcal{T}(t_k - s_k) \mathcal{T}(s_k) \vec{u}_0 - \mathcal{T}(t_k - s_k) \vec{g}_2\|_1 \\ &\leq \|\mathcal{T}(t_k) \vec{u}_0 - \vec{g}_1\|_1 + \|\mathcal{T}(s_k) \vec{u}_0 - \vec{g}_2\|_1, \end{aligned}$$

which tends to 0 as $k \rightarrow \infty$. ■

5 Stationary solutions for the linear molecular motor problem

In this section we show the existence and the uniqueness (up to a multiplicative constant) of the classical stationary solution of the problem for the molecular motor. We suppose that Ω is an open bounded subset of \mathbb{R}^d with smooth boundary $\partial\Omega$.

We consider the linear system

$$\operatorname{div}(\sigma_i \nabla v_i(x) + v_i(x) \nabla \psi_i(x)) + \sum_{j=1}^n a_{ij} v_j(x) = 0 \quad \text{in } \Omega, \quad (5.1)$$

where $i \in \{1, \dots, n\}$, $n > 1$. The system (5.1) is supplemented with the Robin boundary conditions

$$\sigma_i \frac{\partial v_i}{\partial \nu} + v_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (5.2)$$

where $i \in \{1, \dots, n\}$. Thus, the problem can be written as

$$\mathcal{A} \vec{v} = 0,$$

with a linear operator \mathcal{A} in a suitable Banach space \mathcal{X} of functions on Ω , to be made precise later. Moreover, we impose the integral constraint

$$\sum_{i=1}^n \int_{\Omega} v_i(x) \, dx = 1. \quad (5.3)$$

The adjoint problem $\mathcal{A}^* \vec{\varphi} = 0$ to (5.1), in a dual space \mathcal{X}^* , is now

$$\sigma_i \Delta \varphi_i - \nabla \psi_i \cdot \nabla \varphi_i + \sum_{j=1}^n a_{ji} \varphi_j = 0, \quad \text{in } \Omega, \quad (5.4)$$

with the Neumann boundary conditions for each $i = 1, \dots, n$

$$\frac{\partial \varphi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega. \quad (5.5)$$

Since $\sum_{j=1}^n a_{ji} = 0$, the problem (5.4) has the obvious solution

$$\vec{\varphi} = (\varphi_1, \dots, \varphi_n) = (1, \dots, 1). \quad (5.6)$$

We are going to apply the Krein-Rutman theorem on the first eigenvalues and eigenvectors of positive operators, and this will permit us to conclude that the problem (5.1)–(5.2) has a one-dimensional space of solutions. Therefore, under the additional constraint (5.3), the original problem (5.1)–(5.2) has a unique solution.

Perthame and Souganidis sketched this argument for $n > 1$ and $d = 1$ in [20].

Theorem 5.1. *Under the assumption $\sum_{j=1}^n a_{ji} = 0$, there exists a unique smooth solution \vec{v} of the system (5.1)–(5.3).*

Before proving Theorem 5.1 we recall some basic definitions as well as the Krein-Rutman theorem from [9, Ch. VIII, p. 188–191].

Definition 5.2 (Reproducing cone). *We say that a closed set K in \mathcal{X} is a cone, if it possesses the following properties:*

- i) $0 \in K$,
- ii) $u, v \in K \implies \alpha u + \beta v \in K$, for all $\alpha, \beta \geq 0$,
- iii) $v \in K$ and $-v \in K \implies v = 0$.

A cone $K \subset \mathcal{X}$ is said to be reproducing if $\mathcal{X} = K - K \equiv \{k_1 - k_2 : k_1, k_2 \in K\}$.

Definition 5.3 (Dual cone). *If K is a cone in \mathcal{X} , then the set $K^* \subset \mathcal{X}^*$ is said to be a dual cone if*

$$\langle f^*, v \rangle \geq 0,$$

for every $v \in K$.

Definition 5.4 (Strict positivity). *Let \mathcal{B} be a linear operator on \mathcal{X} . Then \mathcal{B} is said to be strongly positive if $\mathcal{B}v \in K^\circ$ for all $v \in K$ such that $v \neq 0$.*

Theorem 5.5. *Let K be a reproducing cone in a Banach space \mathcal{X} , with nonempty interior $K^\circ \neq \emptyset$, and let \mathcal{B} be a strongly positive compact operator on K in a sense of Definition 5.4. Then the spectral radius of \mathcal{B} , $r(\mathcal{B})$, is a simple eigenvalue of \mathcal{B} and \mathcal{B}^* , and their associated eigenvectors belong to K° and $(K^*)^\circ$. More precisely, there exists a unique associated eigenvector in K° (resp. $(K^*)^\circ$) of norm 1. Furthermore, all other eigenvalues are strictly less in absolute value than $r(\mathcal{B})$.*

Proof We will apply Theorem 5.5 to the space $\mathcal{X} = (C(\overline{\Omega}))^n \subset (L^1(\Omega))^n$ endowed with the usual supremum norm, and the operators

$$\begin{aligned}\mathcal{B} &= (\lambda I - \mathcal{A})^{-1} : \mathcal{X} \rightarrow \mathcal{X}, \\ \mathcal{B}^* &= (\lambda I - \mathcal{A}^*)^{-1} : \mathcal{X}^* \rightarrow \mathcal{X}^*,\end{aligned}$$

where $\lambda > 0$ is a strictly positive real number to be fixed later.

Let

$$K = \{\vec{u} \in \mathcal{X} : u_i(x) \geq 0 \text{ for each } x \in \overline{\Omega}, i = 1, \dots, n\}.$$

We remark that K is a reproducing cone, with nonempty interior

$$K^o = \{\vec{u} \in \mathcal{X} : \inf_{x \in \overline{\Omega}} u_i(x) > 0, i = 1, \dots, n\}.$$

From the standard theory [17, Theorem 2.1 and Theorem 3.1, Ch. 7] for elliptic partial differential linear systems, the boundary value problem

$$\sigma_i \Delta \varphi_i - \nabla \psi_i \cdot \nabla \varphi_i + \sum_{j=1}^n a_{ji} \varphi_j - \lambda \varphi_i = f_i \quad \text{in } \Omega, \quad (5.7)$$

with the homogeneous Neumann conditions (5.5) on $\partial\Omega$, for $\lambda = \tilde{\lambda} > 0$ sufficiently large, has a solution $\vec{\varphi} = (\varphi_1, \dots, \varphi_n) \in \mathcal{X}$ for each $\vec{f} = (f_1, \dots, f_n) \in \mathcal{X}$. Moreover, if $f_i(x) \geq 0$ for each $i = 1, \dots, n$, and $x \in \overline{\Omega}$, then $\varphi_i(x) \geq 0$ (in fact, $\varphi_i(x) > 0$ in Ω), which is a consequence of the maximum principle (cf. also Example 3 on p. 196–197 in [9]). Thus, the operator $\mathcal{B}^* = (\tilde{\lambda} I - \mathcal{A}^*)^{-1}$ is a strongly positive and compact operator, and by Theorem 5.5, the largest eigenvalue μ of \mathcal{B} and \mathcal{B}^* is simple.

Since

$$\begin{aligned}-\sigma_i \Delta \varphi_i + \nabla \psi_i \cdot \nabla \varphi_i - \sum_{j=1}^n a_{ji} \varphi_j + \tilde{\lambda} \varphi_i &= \tilde{\lambda} \varphi_i \quad \text{in } \Omega \\ \frac{\partial \varphi_i}{\partial \nu} &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

for all $i \in \{1, \dots, n\}$, with $\vec{\varphi} = (\varphi_1, \dots, \varphi_n) = (1, \dots, 1)$, and since $(1, \dots, 1) \in (K^*)^o$, it follows that $\frac{1}{\tilde{\lambda}} = r\left((\tilde{\lambda} I - \mathcal{A}^*)^{-1}\right)$ is a simple eigenvalue of the operator $(\tilde{\lambda} I - \mathcal{A}^*)^{-1}$. Applying again Theorem 5.5, we deduce that $\frac{1}{\tilde{\lambda}}$ is the largest eigenvalue of the operator $(\tilde{\lambda} I - \mathcal{A})^{-1}$ and that it is simple, and that there exists $\vec{v} \in K^o \subset \mathcal{X}$ such that

$$(\tilde{\lambda} I - \mathcal{A})^{-1} \vec{v} = \frac{1}{\tilde{\lambda}} \vec{v},$$

which is equivalent to

$$\mathcal{A}\vec{v} = 0.$$

This proves the existence of the solution of the problem (5.1)–(5.3). ■

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